A Generalized Algorithm for Computing Matching Polynomials using Determinants

Asa Scherer

April 17, 2007

Abstract

We discuss the connection between determinants of modified matching matrices and the matching polynomial as well as give a recursive algorithm utilizing these determinants to compute the matching polynomial of any graph.

Introduction

This paper assumes that the reader is familiar with the basics of graph theory. In this paper's context, graphs are finite, undirected graphs with no loops or multiple edges.

First, some definitions:

A **circuit cover** of a graph G with vertex set $V(G)$ is a subgraph H of G such that $V(H) = V(G)$ and each connected component of H is a circuit of some order (we define isolated vertices as circuits of order 1 and isolated edges as circuits of order 2.)

A **matching** of a graph is a circuit cover with all circuits having order no greater than 2. A matching can also be thought of as a spanning subgraph with only isolated edges and vertices. If a matching has k edges, that matching is called a k**-matching**.

Let G be a graph with n vertices, vertex set $V(G)$, and edge set $E(G)$. Let m_k be the number of k-matchings of G, and let w_1 and w_2 be complex variables. Then we define

$$M(G) = \sum_{k=0}^{\lfloor n/2 \rfloor} m_k w_1^{n-2k} w_2^k$$

as the **matching polynomial** of G, where $\lfloor n/2 \rfloor$ is the largest integer less than $n/2$. It is easy to see that if a matching has k edges, it must have $n - 2k$ isolated vertices, since each edge contains two vertices. So the matching polynomial is a summation of the product of the terms associated with matching covers (w_1 being associated with vertices and w_2 being associated with edges.) It is worth noting that this is just one definition of the matching polynomial, particularly as defined by E. J. Farrell [1]. An alternate definition will be discussed further on.

Both Farrell and S. A. Wahid have explored many facets of the matching polynomial in detail, in particular linking the matching polynomial to determinants of various matrices associated with graphs. One such matrix is as follows:

Let the **matching matrix** [2] $A(G)$ of a graph G with n vertices be defined as the $n \times n$ matrix with the entry in row i and column j denoted by $[A]_{i,j}$ and

$$[A]_{i,j} = \begin{cases} w_1 & \text{if } i = j \\ \sqrt{w_2} & \text{if } i > j \text{ and } e_{i,j} \in E(G) \\ -\sqrt{w_2} & \text{if } i < j \text{ and } e_{i,j} \in E(G) \\ 0 & \text{if } i \neq j \text{ and } e_{i,j} \notin E(G) \end{cases}$$

Wahid has shown that computing the determinant of this matching matrix yields the matching polynomial for graphs with no even circuits [5]. The question arises: Is it possible to use these determinants to compute *any* graph's matching polynomial? The answer requires a bit more elaboration of Wahid's result, as well as some facts about determinants.

Fact 1 *Let $|A(G)|$ be the determinant of $A(G)$.*[1] *Then $|A(G)|$ can be computed as*

$$|A(G)| = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{1 \le k \le n} [A]_{k,\sigma(k)},$$

where S_n is the set of all permutations on n elements and where $sgn(\sigma) = 1$ if σ is even and $sgn(\sigma) = -1$ if σ is odd [5].

This definition of the determinant as a sum over permutations gives rise to an interesting correllation to circuit covers of G. Any permutation can be written as a product of disjoint permutation cycles. Thus, each permutation of n elements can be thought of as corresponding to a particular circuit cover on $|V(G)| = n$ vertices. Not only that, but there is an additional link between the nonzero terms in the determinant sum of $|A(G)|$ and the circuit covers that exist in G.

Fact 2 *Let σ be a permutation in S_n. Let $b_\sigma = sgn(\sigma) \prod_{1 \le k \le n} [A]_{k,\sigma(k)}$; that is, the term in the determinant sum of $A(G)$ corresponding to σ. Then b_σ is nonzero if and only if the circuit cover corresponding to σ exists in G.*

This is fairly straightforward but tedious to prove, so we will illustrate this concept with an example.

Example:

Let graph G be such that $V(G) = \{1, 2, 3, 4\}$ and $E(G) = \{(1, 2), (2, 3), (3, 4), (4, 1), (1, 3)\}$. (Visually, this can be represented as a rectangle with an edge connecting one set of opposite diagonal vertices.) Then

$$A(G) = \begin{vmatrix} w_1 & \sqrt{w_2} & \sqrt{w_2} & \sqrt{w_2} \\ -\sqrt{w_2} & w_1 & \sqrt{w_2} & 0 \\ -\sqrt{w_2} & -\sqrt{w_2} & w_1 & \sqrt{w_2} \\ -\sqrt{w_2} & 0 & -\sqrt{w_2} & w_1 \end{vmatrix}.$$

Look at permutation $\sigma = (12)(34)$. (Hence, $\sigma(1) = 2$, $\sigma(2) = 1$, $\sigma(3) = 4$, and $\sigma(4) = 3$.) Note that this permutation corresponds to a circuit cover of G with the two isolated edges $(1, 2)$ and $(3, 4)$. Then

$$b_\sigma = sgn(\sigma)[A]_{1,2}[A]_{2,1}[A]_{3,4}[A]_{4,3} = (\sqrt{w_2})(-\sqrt{w_2})(\sqrt{w_2})(-\sqrt{w_2}) = {w_2}^2.$$

Notice that this permutation corresponds to a matching cover of G and its b-term yields the matching term of the cover (${w_2}^2$). In fact, this is true of all permutations corresponding to matching covers and is one of the main components to Wahid's proof of his result.

Now look at permutation $\sigma = (1324)$ (so $\sigma(1) = 3$, $\sigma(2) = 4$, $\sigma(3) = 2$, and $\sigma(4) = 1$), and note that this does not correspond to a circuit cover of G. Then

$$b_\sigma = sgn(\sigma)[A]_{1,3}[A]_{2,4}[A]_{3,2}[A]_{4,1} = -(\sqrt{w_2})(0)(-\sqrt{w_2})(-\sqrt{w_2}) = 0.$$

[1] A notational note: In this paper, vertical lines around a matrix ($|A(G)|$) signify the determinant of that matrix, and vertical lines around a set ($|E(G)|$) signify the cardinality of that set.

So whenever we have a permutation that does not correspond to a circuit cover of G, at least one of the terms in the b_σ product will be 0 since one of the mappings $i \to \sigma(i)$ corresponds to a hypothetical edge that is not there.

Now consider the circuit cover composed of the single circuit of order 4, (1234). This corresponds to permutations $\sigma_1 = (1234)$ and $\sigma_2 = (4321)$. Then

$$b_{\sigma_1} = sgn(\sigma_1)[A]_{1,2}[A]_{2,3}[A]_{3,4}[A]_{4,1} = -(\sqrt{w_2})(\sqrt{w_2})(\sqrt{w_2})(-\sqrt{w_2}) = {w_2}^2$$

and

$$b_{\sigma_2} = sgn(\sigma_2)[A]_{1,4}[A]_{2,1}[A]_{3,2}[A]_{4,3} = -(\sqrt{w_2})(-\sqrt{w_2})(-\sqrt{w_2})(-\sqrt{w_2}) = {w_2}^2.$$

Thus, any permutation σ that corresponds with a circuit cover of G will be such that $b_\sigma \neq 0$. However, we must be careful, as different b-terms in the determinant sum may cancel each other out due to sign difference.

Actually, this cancellation is exactly how Wahid's result arises, as each circuit cover containing proper odd circuits will generate two b-terms, one positive and one negative, and will thus cancel out in the determinant sum, illustrated in the following example:

Consider the circuit cover (123)(4) composed of one circuit of order 3 and one of order 1. This cover corresponds to permutations $\sigma_1 = (123)(4)$ and $\sigma_2 = (321)(4)$. Then

$$b_{\sigma_1} = sgn(\sigma_1)[A]_{1,2}[A]_{2,3}[A]_{3,1}[A]_{4,4} = (\sqrt{w_2})(\sqrt{w_2})(-\sqrt{w_2})(w_1) = -\sqrt{w_2}^3 w_1$$

and

$$b_{\sigma_2} = sgn(\sigma_2)[A]_{1,3}[A]_{2,1}[A]_{3,2}[A]_{4,4} = (\sqrt{w_2})(-\sqrt{w_2})(-\sqrt{w_2})(w_1) = \sqrt{w_2}^3 w_1.$$

But these terms are opposites and will thus cancel when added together in the determinant sum. This cancellation occurs in any permutations associated with a circuit cover containing a proper odd circuit. Hence, the contribution of terms corresponding with circuit covers containing proper odd circuits is negated in the determinant sum. So in a graph with no proper even circuits, the only nonzero terms remaining in the output of the determinant are those corresponding to matching covers, and thus the determinant yields the matching polynomial. However, if our graph has proper even circuits, then we cannot be guaranteed that this is the case; we usually end up with unwanted, uncanceled terms in the determinant sum.

Main Concept

What if we look at a single edge of G? If we "mark" an edge in the matching matrix, what can we determine from the outcome of the determinant? It turns out that, by marking an edge within the matrix, we can keep track of the type of permutations, and thus the type of circuit covers, that it is involved in. But first of all, we look at the types of possible permutations and how they correspond to circuit covers.

Select edge $e = (u, v)$. Partition S_n such that $S_n = S \cup T \cup N$, where

S is the set of permutations σ in S_n such that $\sigma(u) = v$ and $\sigma(v) = u$,

T is the set of permutations σ in S_n such that either $\sigma(u) = v$ and $\sigma(v) \neq u$ or $\sigma(u) \neq v$ and $\sigma(v) = u$,

and N is the set of permutations σ in S_n such that $\sigma(u) \neq v$ and $\sigma(v) \neq u$.

Expressing σ as a product of disjoint cycles yields the following structures:

If $\sigma \in S$, then $\sigma = (uv)t$, where t is a permutation on the $n-2$ elements of $V(G) - \{u, v\}$. Thus, σ corresponds to a circuit cover of G that contains $e = (u, v)$ as an isolated edge.

If $\sigma \in T$, then $\sigma = (uvw_1 \dots w_k)p$, $k \geq 1$ or $\sigma = (vuw_1 \dots w_k)p$, $k \geq 1$, where p is a permutation on the elements not given in the first cycle and w_i is some vertex in $V(G)$. Thus, σ corresponds to a circuit cover of G that contains e as part of a proper circuit.

If $\sigma \in N$, then $\sigma = c_1c_2 \dots c_k$, where the c_i are disjoint cycles and, for each $1 \leq i \leq k$, $c_i \neq (uv)$, $c_i \neq (uv \dots l)$, and $c_i \neq (vu \dots l)$. Thus, σ corresponds to a circuit cover of G that does not include e as an edge in the cover.

The advantage of distinguishing these permutations is that we *know* that, when σ is in T, the corresponding term in the determinant sum does not contribute a valid term to the matching polynomial, as these permutations contain proper cycles (which, in turn, correspond to covers with proper circuits.) The trick is, how do we differentiate between terms in the output of the determinant? That is, how do we know particular terms arose from permutations from S, T, or N?

Marked Matrix

Given graph G with matching matrix $A(G)$, choose an edge $e = (u, v) \in E(G)$ and call it the **marked edge of** G. Let the **marked matrix** of G with marked edge $e = (u, v)$ and complex number a be denoted $A_e(G, a)$, with the entry in row i and column j denoted by $[A_e]_{i,j}$. Define this matrix by

$[A_e]_{u,v} = a[A]_{u,v}$

$[A_e]_{v,u} = a[A]_{v,u}$

$[A_e]_{i,j} = [A]_{i,j}$ if $(i, j) \neq (u, v)$ and $(i, j) \neq (v, u)$.

So, essentially, the marked matrix is $A(G)$ with $[A]_{u,v}$ and $[A]_{v,u}$ "marked by" a. The following theorem lets us use the marked matrix to distinguish from which permutation set (S, T, or N) a particular term in the determinant output arose. We will break part of the theorem into three lemmas:

Given a graph with $|V(G)| = n$ and a marked matrix $A_e(G, a)$, define $b_\sigma = sgn(\sigma) \prod_{1 \leq k \leq n} [A_e]_{k,\sigma(k)}$, where $\sigma \in S_n$.

Lemma 1 *If $\sigma \in S$, then $b_\sigma = \pm a^2 w_1^p \sqrt{w_2}^q$ for some integers p and q.*

Proof

Since $\sigma \in S$, then $\sigma(u) = v$ and $\sigma(v) = u$.

So $b_\sigma = sgn(\sigma) \prod_{1 \leq k \leq n} [A_e]_{k,\sigma(k)}$

$$
\begin{aligned}
&= sgn(\sigma)[A_e]_{u,\sigma(u)}[A_e]_{v,\sigma(v)} \prod_{\substack{t \neq u,v \\ 1 \leq t \leq n}} [A_e]_{k,\sigma(k)} \\
&= sgn(\sigma)[A_e]_{u,v}[A_e]_{v,u} \prod_{\substack{t \neq u,v \\ 1 \leq t \leq n}} [A_e]_{k,\sigma(k)} \\
&= sgn(\sigma)a[A]_{u,v}a[A]_{v,u)} \prod_{\substack{t \neq u,v \\ 1 \leq t \leq n}} [A]_{t,\sigma(t)} \quad \text{(By definition of the marked matrix)} \\
&= sgn(\sigma)a^2[A]_{u,v}[A]_{v,u)} \prod_{\substack{t \neq u,v \\ 1 \leq t \leq n}} [A]_{t,\sigma(t)} \\
&= \pm a^2 w_1^p \sqrt{w_2}^q.
\end{aligned}
$$

We can conclude this because the only entries of $A(G)$ are w_1, $\sqrt{w_2}$, and $-\sqrt{w_2}$.

■

Lemma 2 *If $\sigma \in T$, then $b_\sigma = \pm a w_1^p \sqrt{w_2}^q$ for some integers p and q.*

Proof

Since $\sigma \in T$, then $\sigma(u) = v$ and $\sigma(v) \neq u$, or $\sigma(u) \neq v$ and $\sigma(v) = u$.
Without loss of generality, suppose $\sigma(u) = v$ and $\sigma(v) \neq u$.
Then $b_\sigma = sgn(\sigma) \prod_{1 \leq k \leq n} [A_e]_{k,\sigma(k)}$

$$
\begin{aligned}
&= sgn(\sigma)[A_e]_{u,\sigma(u)}[A_e]_{v,\sigma(v)} \prod_{\substack{t \neq u,v \\ 1 \leq t \leq n}} [A_e]_{k,\sigma(k)} \\
&= sgn(\sigma)[A_e]_{u,v}[A_e]_{v,\sigma(v)\neq u} \prod_{\substack{t \neq u,v \\ 1 \leq t \leq n}} [A_e]_{k,\sigma(k)} \\
&= sgn(\sigma)a[A]_{u,v}[A]_{v,\sigma(v)\neq u} \prod_{\substack{t \neq u,v \\ 1 \leq t \leq n}} [A]_{t,\sigma(t)} \quad \text{(By definition of the marked matrix)} \\
&= \pm a w_1^p \sqrt{w_2}^q.
\end{aligned}
$$

Similar to the lemma above, we can conclude this because the only entries of $A(G)$ are w_1, $\sqrt{w_2}$, and $-\sqrt{w_2}$.

■

Lemma 3 *If $\sigma \in N$, then $b_\sigma = \pm w_1^p \sqrt{w_2}^q$ for some integers p and q.*

Proof

Since $\sigma \in N$, then $\sigma(u) \neq v$ and $\sigma(v) \neq u$.
Then $b_\sigma = sgn(\sigma) \prod_{1 \leq k \leq n} [A_e]_{k,\sigma(k)}$.
$(u, \sigma(u)) \neq (u, v)$ and $(v, \sigma(v)) \neq (v, u)$, so
$b_\sigma = sgn(\sigma) \prod_{1 \leq k \leq n} [A]_{k,\sigma(k)}$ (Definition of the marked matrix.)
$= \pm w_1^p \sqrt{w_2}^q$. Again, we can conclude this because the only entries of $A(G)$ are w_1, $\sqrt{w_2}$, and $-\sqrt{w_2}$.

■

Now we can prove our theorem:

Theorem 1 *Given graph G and marked matrix $A_e(G,a)$ with marked edge $e=(u,v)$, let b_σ be the term in $|A_e(G,a)|$ corresponding to permutation σ. Then*

$\sigma \in S$ *if and only if* $b_\sigma = \pm a^2 w_1^p \sqrt{w_2}^q$ *for some non-negative integers p and q.*

$\sigma \in T$ *if and only if* $b_\sigma = \pm a w_1^p \sqrt{w_2}^q$ *for some non-negative integers p and q.*

$\sigma \in N$ *if and only if* $b_\sigma = \pm w_1^p \sqrt{w_2}^q$ *for some non-negative integers p and q.*

Proof

The above three lemmas prove that one way of each implication holds. And because there are only three possibilities for σ, that is, $\sigma \in S$, $\sigma \in T$, or $\sigma \in N$, and the union of these three sets is S_n, we can make each of the previous lemmas into an if and only if.

■

Fundamental Marked Matrix Theorem

With these results, we can now formulate a theorem that will form the basis for our matching polynomial algorithm. Once again, we will split up this theorem in the form of three lemmas, each dealing with a different permutation type. But first we must state some basic assertions and hypotheses.

The following assumptions apply to the three following lemmas as well as the Fundamental Theorem:

Given graph G with $|E(G)| \geq 1$ and $|V(G)| \geq 3$, matching matrix $A(G)$ and marked matrix $A_e(G,a)$ with marked edge $e=(u,v)$, define subgraphs $G' = G - \{e\}$ and $G'' = G - \{u,v\}$ (based on the marked edge.) (We give the stipulation that $|V(G)| \geq 3$ so that we can build a non-null matching matrix of G''.)

Now again define $b_\sigma = sgn(\sigma)\prod_{1\leq k\leq n}[A_e]_{k,\sigma(k)}$, where $\sigma \in S_n$. Then

$|A_e(G,a)| = \sum_{\sigma\in S} b_\sigma + \sum_{\sigma \in T} b_\sigma + \sum_{\sigma\in N} b_\sigma$. (This follows from T, S, and N forming a partition on S_n.)

Also, $|A_e(G,1)| = |A(G)|$ (This follows from the fact that $A_e(G,1) = A(G)$ by the definition of the marked matrix.)

Lemma 4 *If $a = 1$ so that $A_e(G,a) = A_e(G,1) = A(G)$, then $\sum_{\sigma\in S} b_\sigma = w_2|A(G'')|$.*

Proof

We know that $\sum_{\sigma\in S} b_\sigma = \sum_{\sigma\in S} sgn(\sigma)\prod_{1\leq k\leq n}[A_e]_{k,\sigma(k)}$.

All σ in S are of the form $\sigma = (uv)t$ where t is a permutation on the $n-2$ elements of $V(G) - \{u,v\}$. Let R be the set of all permutations t on the set $V(G) - \{u,v\}$. It is easy to see that $\{(uv)t | t \in R\} = S$. Note that $sgn(\sigma) = -sgn(t)$ for each t corresponding to $\sigma \in S$.

Let t be such a permutation on the $n-2$ elements of $V(G) - \{u,v\}$ so that $\sigma = (uv)t$. Then

$\sum_{\sigma\in S} b_\sigma = \sum_{t\in R} sgn(\sigma)[A_e]_{u,\sigma(u)}[A_e]_{v,\sigma(v)} \prod_{\substack{k\neq u,v \\ 1\leq k\leq n}} [A_e]_{k,\sigma(k)}$

$= \sum_{t\in R} sgn(\sigma)[A_e]_{u,v}[A_e]_{v,u} \prod_{\substack{k\neq u,v \\ 1\leq k\leq n}} [A]_{k,t(k)}$

$= \sum_{t\in R} sgn(\sigma) a\sqrt{w_2}(-a\sqrt{w_2}) \prod_{\substack{k\neq u,v \\ 1\leq k\leq n}} [A]_{k,t(k)}$

$= \sum_{t\in R} sgn(\sigma)(-a^2 w_2) \prod_{\substack{k \neq u,v \\ 1 \leq k \leq n}} [A]_{k,t(k)}$

Using $a = 1$ so that $A_e(G, a) = A_e(G, 1) = A(G)$, we have, for $|A(G)|$,

$\sum_{\sigma\in S} b_\sigma = w_2 \sum_{t\in R} sgn(t) \prod_{\substack{k \neq u,v \\ 1 \leq k \leq n}} [A]_{k,t(k)}$.

But $\sum_{t\in R} sgn(t) \prod_{\substack{k \neq u,v \\ 1 \leq k \leq n}} [A]_{k,t(k)} = |A(G - \{u, v\})| = |A(G'')|$.

So $w_2 \sum_{t\in R} sgn(t) \prod_{\substack{k \neq u,v \\ 1 \leq k \leq n}} [A]_{k,t(k)} = w_2|A(G'')|$.

Thus, for $|A(G)| = |A_e(G, 1)|$,

$$\sum_{\sigma\in S} b_\sigma = w_2|A(G'')|.$$

■

Lemma 5 *For* $|A(G)| = |A_e(G, 1)|$, $\sum_{\sigma\in N} b_\sigma = |A(G')|$.

Proof

By Theorem 1, every term of $\sum_{\sigma\in S} b_\sigma$ for $A_e(G, a)$ has the form $\pm a^2 w_1^p \sqrt{w_2}^q$ for some integers p and q and every term of $\sum_{\sigma\in T} b_\sigma$ has the form $\pm a w_1^l \sqrt{w_2}^m$ for some integers l and m. Thus, setting $a = 0$ will result in

$\sum_{\sigma\in S} b_\sigma = 0$ and

$\sum_{\sigma\in T} b_\sigma = 0$.

Since $|A_e(G, a)| = \sum_{\sigma\in S} b_\sigma + \sum_{\sigma\in T} b_\sigma + \sum_{\sigma\in N} b_\sigma$,

$|A_e(G, 0)| = 0 + 0 + \sum_{\sigma\in N} b_\sigma = \sum_{\sigma\in N} b_\sigma$.

But $A_e(G, 0) = A(G - \{e\})$ by the structure of $A_e(G, a)$.

So $|A_e(G, 0)| = |A(G - \{e\})| = |A(G')|$.

Note that, since terms of $\sum_{\sigma\in N} b_\sigma$ have the form $\pm w_1^p \sqrt{w_2}^q$ for some integers p and q, then $\sum_{\sigma\in N} b_\sigma$ for $|A_e(G, 1)| = |A(G)|$ is the same as $\sum_{\sigma\in N} b_\sigma$ for $|A_e(G, 0)|$.

Thus, for $|A(G)|$,

$$\sum_{\sigma\in N} b_\sigma = |A(G')|.$$

■

Lemma 6 $\sum_{\sigma\in T} b_\sigma = Im(|A_e(G, i)|)$.

Proof

Again, every term of $\sum_{\sigma\in S} b_\sigma$ has the form $\pm a^2 w_1^p \sqrt{w_2}^q$ for some integers p and q, every term of $\sum_{\sigma\in T} b_\sigma$ has the form $\pm a w_1^l \sqrt{w_2}^m$ for some integers l and m, and every term of $\sum_{\sigma\in N} b_\sigma$ has the form $\pm w_1^r \sqrt{w_2}^s$ for some integers r and s.

Now set $a = i$ and look at $|A_e(G, i)| = \sum_{\sigma\in S} b_\sigma + \sum_{\sigma\in T} b_\sigma + \sum_{\sigma\in N} b_\sigma$. The terms of $\sum_{\sigma\in N}$ will not have any imaginary component, as they do not contain any a's. The terms of $\sum_{\sigma\in S} b_\sigma$ will simply gain an $i^2 = -1$ in place of the a^2. Finally, the terms of $\sum_{\sigma\in T} b_\sigma$ will all gain an i in place of the a. Thus, the only imaginary components of $|A_e(G, i)|$ will arise from $\sum_{\sigma\in T} b_\sigma$, and every term in $\sum_{\sigma\in T} b_\sigma$ will be imaginary. Thus, for $|A_e(G, 1)| = |A(G)|$,

$$\sum_{\sigma\in T} b_\sigma = Im(|A_e(G, i)|).$$

■

Theorem 2 *Given graph G with $|E(G)| \geq 1$ and $|V(G)| \geq 3$, matching matrix $A(G)$ and marked matrix $A_e(G, a)$ with marked edge $e = (u, v)$, define subgraphs $G' = G - \{e\}$ and $G'' = G - \{u, v\}$ (based on the marked edge.) Then*

$$|A(G)| = w_2|A(G'')| + |A(G')| + Im(|A_e(G, i)|)$$

Proof

We already know that $|A_e(G, 1)| = \sum_{\sigma \in S} b_\sigma + \sum_{\sigma \in T} b_\sigma + \sum_{\sigma \in N} b_\sigma$, and that $|A_e(G, 1)| = |A(G)|$.

Hence, with this information and the above lemmas,

$$|A_e(G, 1)| = \sum_{\sigma \in S} b_\sigma + \sum_{\sigma \in T} b_\sigma + \sum_{\sigma \in N} b_\sigma$$

can be rewritten as

$$|A(G)| = w_2|A(G'')| + |A(G')| + Im(|A_e(G, i)|).$$

■

Thus, we can now break down the determinant of a matching matrix into determinants of subgraphs. This result is very similar to the Fundamental Edge Theorem of Matching Theory, which states, given a graph G and subgraphs G' and G'' as previously defined, that $|M(G)| = w_2|M(G'')| + |M(G')|$ [1]. We will use this basic idea in the following algorithm.

Algorithm

Now that we've set up the necessary concepts and theorems, we can address the main goal of this paper: constructing the matching polynomial of a generalized graph using determinants.

Theorem 3 *Let G be a graph with $|E(G)| \geq 1$. For each subgraph H of G with $|E(H)| \geq 1$, choose one edge (in H) to be associated with H. Call this edge the* **splitting edge of H** *and denote it e_H. Let B_G be a set of subgraphs of G, constructed in the following way:*

Add G to B_G. If $|A(G)| = M(G)$, we are done. If not, let $e_G = (u, v)$ be the splitting edge of G. Create subgraphs $G'' = G - \{u, v\}$ and $G' = G - \{e_G\}$, adding both to B_G. Repeat this process for the resulting subgraphs until $|A(H)| = M(H)$, in which case do not split H and do not add H to B_G.

Then

$$M(G) = |A(G)| - \sum_{\Gamma \in B_G} w_2^j Im(|A_{e_\Gamma}(\Gamma, i)|),$$

where e_Γ is the splitting edge of Γ and $j = \frac{1}{2}|V(G) - V(\Gamma)|$.

Proof

We will perform a proof by induction on the number of edges in G.

Base Case

Let $1 \leq |E(G)| \leq 4$. Assign G a set of subgraphs B_G in the manner described above.

Case 1: $|E(G)| \leq 3$. Then G cannot contain any proper even circuits (order 4 or greater) and hence $|A(G)| = M(G)$ by Wahid's previously stated result. So $B_G = \{G\}$. Let e_G be the splitting edge of G. We know from a previous lemma that $Im(|A_{e_G}(G,i)) = \sum_{\sigma \in T} b_\sigma$ for $|A(G)|$. But permutations $\sigma \in T$ correspond to circuit covers of G containing proper circuits. Since we know that the total contribution of proper odd circuits is 0 in $|A(G)|$ and G contains no proper even circuits, then by Fact 2, $Im(|A_{e_G}(G,i)) = 0$. Hence, $M(G) = |A(G)| - Im(|A_{e_G}(G,i))$ and our theorem holds.

Case 2: $|E(G)| = 4$. Then the creation of B_G looks like this: Add G to B_G. Let $e_G = (u,v)$ be the splitting edge of G. Create G' and G''. Then $|E(G')| \leq 3$ and $|E(G'')| \leq 3$. So G' and G'' cannot contain any proper even circuits. Thus, by Wahid's previously stated result, $|A(G')| = M(G')$ and $|A(G'')| = M(G'')$. So G' is not in B_G and G'' is not in B_G and neither G' nor G'' is split any further. Thus, $B_G = \{G\}$.

Now we use our fundamental theorem:

Let e_G be our marked edge. Then

$|A(G)| = w_2|A(G'')| + |A(G')| + Im(|A_{e_G}(G,i)|)$

$= w_2M(G'') + M(G') + Im(|A_{e_G}(G,i)|)$

$= M(G) + Im(|A_{e_G}(G,i)|)$ (By the Fundamental Edge Theorem of Matching Theory)

$= M(G) + \sum_{\Gamma \in B_G} w_2^j Im(|A_{e_\Gamma}(\Gamma,i)|)$. $(j = \frac{1}{2}|V(G) - V(G)| = 0)$.

Thus, the theorem holds for $1 \leq |E(G)| \leq 4$.

Inductive Step

Suppose this theorem holds for $|E(G)| \leq k$.

Let $|E(G)| = k+1$. Pick a splitting edge for each subgraph of G with at least one edge and construct B_G.

Look at

$$|A(G)| = w_2|A(G'')| + |A(G')| + Im(|A_{e_G}(G,i)|),$$

where $e_G = (u,v)$ is the splitting edge of G.

Construct $B_{G'}$ and $B_{G''}$ using the splitting edges of each subgraph already assigned. Then $B_{G'}$ is the set of subgraphs of G' that are in B_G and $B_{G''}$ is the set of subgraphs of G'' that are in B_G. So $B_{G'} \cup B_{G''} = B_G - \{G\}$.

We know that $|E(G')| \leq k$ and $|E(G'')| \leq k$ by the construction of G' and G''. So the theorem holds for both G' and G''. Thus,

$$|A(G)| = w_2 \left(M(G'') + \sum_{\Gamma \in B_{G''}} \left(w_2^{j''} Im\left(|A_{e_\Gamma}(\Gamma,i)|\right)\right)\right)$$

$$+ \left(M(G') + \sum_{\Gamma \in B_{G'}} \left(w_2^{j'} Im\left(|A_{e_\Gamma}(\Gamma,i)|\right)\right)\right) + Im\left(|A_{e_G}(G,i)|\right),$$

where $j' = \frac{1}{2}|V(G') - V(\Gamma)|$ for any $\Gamma \in B_{G'}$ and $j'' = \frac{1}{2}|V(G'') - V(\Gamma)|$ for $\Gamma \in B_{G''}$.

We can rearrange this expression to

$$|A(G)| = w_2 M(G'') + M(G') + w_2 \sum_{\Gamma \in B_{G''}} w_2^{j''} Im(|A_{e_\Gamma}(\Gamma, i)|)) + \sum_{\Gamma \in B_{G'}} w_2^{j'} Im(|A_{e_\Gamma}(\Gamma, i)|) + Im(|A_{e_G}(G, i)|)$$

Hence,

$$|A(G)| = M(G) + w_2 \sum_{\Gamma \in B_{G''}} w_2^{j''} Im(|A_{e_\Gamma}(\Gamma, i)|) + \sum_{\Gamma \in B_{G'}} w_2^{j'} Im(|A_{e_\Gamma}(\Gamma, i)|) + Im(|A_{e_G}(G, i)|)$$

by the Fundamental Edge Theorem of Matching Theory.

Now note

$$w_2 \sum_{\Gamma \in B_{G''}} w_2^{j''} Im(|A_{e_\Gamma}(\Gamma, i)|) = \sum_{\Gamma \in B_{G''}} w_2^{j''+1} Im(|A_{e_\Gamma}(\Gamma, i)|).$$

Let $j = \frac{1}{2}|V(G) - V(\Gamma)|$, where Γ is a subgraph of G. Suppose Γ is a subgraph of G''. $\frac{1}{2}|V(G) - V(G'')| = 1$, so $j = \frac{1}{2}|V(G) - V(G'')| + \frac{1}{2}|V(G'') - V(\Gamma)| = 1 + j''$.

Suppose Γ is a subgraph of G'. $\frac{1}{2}|V(G) - V(G')| = 0$, so $j = \frac{1}{2}|V(G) - V(G')| + \frac{1}{2}|V(G') - V(\Gamma)| = 0 + j' = j'$.

Taking the fact that $B_{G'} \cup B_{G''} = B_G - \{G\}$ into account, we can combine the sums, obtaining

$$w_2 \sum_{\Gamma \in B_{G''}} w_2^{j''} Im(|A_{e_\Gamma}(\Gamma, i)|) + \sum_{\Gamma \in B_{G'}} w_2^{j'} Im(|A_{e_\Gamma}(\Gamma, i)|) = \sum_{\Gamma \in B_G - \{G\}} w_2^{j} Im(|A_{e_\Gamma}(\Gamma, i)|).$$

So

$$|A(G)| = M(G) + \sum_{\Gamma \in B_G - \{G\}} w_2^{j} Im(|A_{e_\Gamma}(\Gamma, i)|) + Im(|A_{e_G}(G, i)|)$$

.

We can include the last term in our sum if we add G to the set $B_G - \{G\}$, since $\frac{1}{2}|V(G) - V(G)| = 0$:

$$|A(G)| = M(G) + \sum_{\Gamma \in B_G} w_2^{j} Im(|A_{e_\Gamma}(\Gamma, i)|)$$

Thus,

$$M(G) = |A(G)| - \sum_{\Gamma \in B_G} w_2^{j} Im(|A_{e_\Gamma}(\Gamma, i)|)$$

for graphs with $|E(G)| = k + 1$.

Hence, by the principal of mathematical induction, this theorem holds for any graph with at least one edge.

■

Conceptual Discussion

The main purpose of this algorithm is to counteract the effect of circuit covers with proper even circuits on the determinant of the matching matrix. $|A(G)|$ will equal the matching polynomial plus the contributions of circuit covers with proper even circuits (Farrell and Wahid's construction of $A(G)$ guarantees that terms corresponding to covers with proper odd circuits will not contribute to $|A(G)|$) [5]. This algorithm computes the exact contribution of these unwanted covers (computed by finding the imaginary component of the marked matrix determinant with $a = i$), and subtracts them from $|A(G)|$ to reach $M(G)$.

The construction of B_G was left intentionally non-specific in cutting off when subgraphs H are such that $|A(H)| = M(H)$ so that different implementations may be created from this theorem. For instance, one implementation could simply remove edges until the resulting subgraphs had 4 edges or less. Alternately, if certain information is known about the graph, for example, which set of edges must be removed to create a spanning tree, then another implementation could remove just those edges in the splitting edge process. Once all these edges are removed, we know that the "base case" $|A(H)| = M(H)$ must hold (since no circuits remain.)

It was mildly hoped that this general algorithm would be more efficient than the standard recursive algorithms for computing the matching polynomial. Brief estimations on computational efficiency, however, show otherwise, with the algorithm having somewhere around factorial complexity in a general case. In particular cases, though, where we know a bit about the graph and there are a small number of edges that, if taken away, get rid of all the proper even circuits in the graph, there are obvious advantages to this algorithm.

Further Conclusions

Though using the matching matrix as Farrell and Wahid define it is quite valuable in cutting down complexity through the cancellation of terms relating to odd circuits, it is worth proving that this algorithm will work even if we use a version of the matching matrix that has no negative numbers in it; that is, if every term that was $-\sqrt{w_2}$ in the matching matrix becomes $\sqrt{w_2}$. Let us define this matrix as the **positive matching matrix of** G, denoted $A_+(G)$, in the following way:

$$[A_+]_{i,j} = \begin{cases} w_1 & \text{if } i = j \\ \sqrt{w_2} & \text{if } i > j \text{ and } e_{i,j} \in E(G) \\ \sqrt{w_2} & \text{if } i < j \text{ and } e_{i,j} \in E(G) \\ 0 & \text{if } i \neq j \text{ and } e_{i,j} \notin E(G) \end{cases}$$

This proof only requires a few alterations from the one above, the main difference being that we may get some negative coefficients on our matching terms as an end result of the determinant. However, the coefficients will be correct (that is, match up with our definition of the matching polynomial) except for the negative signs. Let us define the **alternating matching polynomial** $M_-(G)$ as

$$M_-(G) = \sum_{k=0}^{n/2} (-1)^k m_k w_1^{n-2k} w_2^k.$$

(In fact, this is actually one of the alternate definitions of the matching polynomial referred to earlier, almost identical to that given by C. D. Godsil) [4].

First, we prove that $|A_+(G)| = M_-(G)$ for graphs with no proper circuits:

Theorem 4 *Suppose G is a graph with no proper circuits. Then $|A_+(G)| = M_-(G)$.*

Proof

$$|A_+(G)| = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{1 \le k \le n} [A_+]_{k,\sigma(k)}$$

Let b_σ be the term in this sum arising from the permutation σ.

Since there are no proper circuits in G, there will be no terms in $|A_+(G)|$ corresponding to a permutation with proper cycles. Hence, the only nonzero terms in $|A_+(G)|$ will correspond with matching covers of G. Consider a particular matching cover type, say a k-matching. These matchings will correspond with permutations in S_n that can be expressed as the product of k transpositions.

Suppose k is even. Let σ correspond to a k-matching. Then σ is an even permutation, as it can be expressed as the product of k disjoint transpositions. Then $b_\sigma = w_1^{n-2k} w_2^k$, since $sgn(\sigma) = 1$. Hence, when we add up all such terms corresponding to k-matchings, we will get $m_k w_1^{n-2k} w_2^k = (-1)^k m_k w_1^{n-2k} w_2^k$.

Now suppose k is odd. Let σ correspond to a k-matching. Then σ is an odd permutation, as it can be expressed as the product of k disjoint transpositions. Then $b_\sigma = -w_1^{n-2k} w_2^k$, since $sgn(\sigma) = -1$. Hence, when we add up all such terms corresponding to k-matchings, we will get $-m_k w_1^{n-2k} w_2^k = (-1)^k m_k w_1^{n-2k} w_2^k$.

Thus, $|A_+(G)| = M_-(G)$.

■

The main idea here is that we cannot have the same type of matching cover generating different sgn's, so the resulting coefficients in $|A_+(G)|$ will be consistent with the matching polynomial, with the additional structure that negative signs on matchings correspond with odd permutations.

As for the marked version of $|A_+(G)|$, we define it exactly the same way as before: Choose an edge $e = (u, v) \in E(G)$ and call it the **marked edge of** G. Let the **positive marked matrix** of G with marked edge $e = (u, v)$ and complex number a be denoted $A_{+e}(G, a)$, with the entry in row i and column j denoted by $[A_{+e}]_{i,j}$. Define this matrix by

$[A_{+e}]_{u,v} = a[A_+]_{u,v}$

$[A_{+e}]_{v,u} = a[A_+]_{v,u}$

$[A_{+e}]_{i,j} = [A_+]_{i,j}$ if $(i, j) \neq (u, v)$ and $(i, j) \neq (v, u)$.

It follows from this definition that a slightly altered version of the Fundamental Marked Matrix Theorem still holds. That is,

Theorem 5 *Given graph G with $|E(G)| \ge 1$ and $V(G) \ge 3$, positive matching matrix $A_+(G)$ and positive marked matrix $A_{+e}(G, a)$ with marked edge $e = (u, v)$, define subgraphs $G' = G - \{e\}$ and $G'' = G - \{u, v\}$ (based on the marked edge.) Then*

$$|A_+(G)| = -w_2|A_+(G'')| + |A_+(G')| + Im(|A_{+e}(G, i)|)$$

First, it is clear that Theorem 1 still holds, since the only change we've made is to signs, which are not a factor in the conclusion of the theorem. Using this result, two parts of this theorem's proof are identical to the unaltered version:

Proving that $\sum_{\sigma\in N} b_\sigma = |A_+(G')|$ goes exactly the same way, as all we have to do is show that $A_{+e}(G,0) = \sum_{\sigma\in N} b_\sigma$. Since the b terms corresponding with permutations from both S and T all have a's multiplying them, they drop out when we substitute 0 for a.

Similarly, the proof for $\sum_{\sigma\in T} b_\sigma = Im(|A_{+e}(G,i)|)$ only requres the basic form of each b term (given by our Theorem 1), so it remains the same.

The altered portion of the proof is as follows:

Lemma 7 *If $a = 1$ so that $A_{+e}(G,a) = A_{+e}(G,1) = A_+(G)$, then $\sum_{\sigma\in S} b_\sigma = -w_2|A_+(G'')|$*

Proof

We know that $\sum_{\sigma\in S} b_\sigma = \sum_{\sigma\in S} sgn(\sigma)\prod_{1\le k\le n}[A_e]_{k,\sigma(k)}$

All σ in S are of the form $\sigma = (uv)t$ where t is a permutation on the $n-2$ elements of $V(G) - \{u,v\}$. Let R be the set of all permutations t on the set $V(G) - \{u,v\}$. It is easy to see that $\{(uv)t|t \in R\} = S$. Note that $sgn(\sigma) = -sgn(t)$ for each t corresponding to $\sigma \in S$.

Let t be such a permutation on the $n-2$ elements of $V(G) - \{u,v\}$ so that $\sigma = (uv)t$. Then

$\sum_{\sigma\in S} b_\sigma = \sum_{t\in R} sgn(\sigma)[A_{+e}]_{u,\sigma(u)}[A_{+e}]_{v,\sigma(v)}\prod_{\substack{k\neq u,v\\1\le k\le n}}[A_{+e}]_{k,\sigma(k)}$

$= \sum_{t\in R} sgn(\sigma)[A_{+e}]_{u,v}[A_{+e}]_{v,u}\prod_{\substack{k\neq u,v\\1\le k\le n}}[A_+]_{k,t(k)}$

$= \sum_{t\in R} sgn(\sigma)a\sqrt{w_2}(a\sqrt{w_2})\prod_{\substack{k\neq u,v\\1\le k\le n}}[A_+]_{k,t(k)}$

$= \sum_{t\in R} sgn(\sigma)(a^2w_2)\prod_{\substack{k\neq u,v\\1\le k\le n}}[A_+]_{k,t(k)}$

Using $a = 1$ so that $A_{+e}(G,a) = A_{+e}(G,1) = A(G)$, we have, for $|A(G)|$,

$\sum_{\sigma\in S} b_\sigma = w_2\sum_{t\in R}(-sgn(t))\prod_{\substack{k\neq u,v\\1\le k\le n}}[A_+]_{k,t(k)}$

But $\sum_{t\in R}(-sgn(t))\prod_{\substack{k\neq u,v\\1\le k\le n}}[A_+]_{k,t(k)} = -|A_+(G-\{u,v\})| = -|A_+(G'')|$.

So $w_2\sum_{t\in R}(-sgn(t))\prod_{\substack{k\neq u,v\\1\le k\le n}}[A_+]_{k,t(k)} = -w_2|A_+(G'')|$.

Thus, for $|A_+(G)| = |A_{+e}(G,1)|$,

$$\sum_{\sigma\in S} b_\sigma = -w_2|A(G'')|.$$

■

We need one more result in order to make our main algorithm work with $A_+(G)$, and that is an altered version of the fundamental edge theorem of matching theory:

Theorem 6 *Let G be a graph with $|E(G)| \ge 1$. Pick an edge $e \in E(G)$, and define subgraphs $G' = G - \{e\}$ and $G'' = G - \{u,v\}$. Then*

$$M_-(G) = -w_2M_-(G'') + M_-(G')$$

Proof

Essentially, $M_-(G')$ adds up all the matching terms of $M_-(G)$ for matchings that do not include edge e.

Similarly, $w_2 M_-(G'')$ counts all the matching terms of $M_-(G)$ for matchings that *do* include edge e. But since G'' has one fewer edge than G, the sign on each matching term will be swapped. Thus, we have to multiply $w_2 M_-(G'')$ by -1 to have it correspond with $M_-(G)$. The set of matchings including edge e and the set of matchings that do not include edge e make up a partition on all the matchings of G. Hence,

$$M_-(G) = -w_2 M_-(G'') + M_-(G')$$

■

Now the only thing left is to modify the main algorithm:

Theorem 7 *Let G be a graph with $|E(G)| \geq 1$. For each subgraph H of G with $|E(H)| \geq 1$, choose one edge (in H) to be associated with H. Call this edge the splitting edge of H and denote it e_H. Let B_G be a set of subgraphs of G, constructed in the following way:*

Add G to B_G. Let $e_G = (u, v)$ be the splitting edge of G. Create subgraphs $G'' = G - \{u, v\}$ and $G' = G - \{e_G\}$, adding both to B_G. Repeat this process for the resulting subgraphs until $|A_+(H)| = M_-(H)$, in which case do not split H and do not add H to B_G.

Then

$$M_-(G) = |A_+(G)| - \sum_{\Gamma \in B_G} (-w_2)^j Im(|A_{+e_\Gamma}(\Gamma, i)|),$$

where e_Γ is the splitting edge of Γ and $j = \frac{1}{2}|V(G) - V(\Gamma)|$.

This proof will mostly go the same way it did before, with the occasional substitutions of $A_+(G)$ and $-w_2$ for w_2. The main difference is in the base case, but that is simple enough. If we change the base case from $1 \leq |E(G)| \leq 4$ to $1 \leq |E(G)| \leq 3$, it is easy to show that the modified theorem holds.

Thus, we have generalized our algorithm even further to encompass an alternate definition of matching matrix and of matching polynomial.

Related Algorithms

Our proven algorithm essentially marks one edge at a time to compute unwanted determinant terms and then breaks the graph down in order to compute yet more unwanted terms until all the terms are accounted for. Professor Mike Spivey of the University of Puget Sound suggested to me the idea of marking *all* edges at once and taking the determinant of the matching matrix. That is, if $E(G) = m$, construct m variables i_k such that ${i_k}^2 = -1$ for all $1 \leq k \leq m$ and such that each i_k is associated with one edge. Define matrix $O(G)$ by

$[O(G)]_{u,v} = i_k [A]_{u,v}$ if edge $(u, v) \in E(G)$, where i_k is the variable associated with edge (u, v)

$[O(G)]_{u,v} = 0$ if edge (u, v) is not in $E(G)$.

Hence, we have essentially "marked" every edge with a different variable. Now if we take the determinant of $O(G)$ and subtract off the "imaginary" terms (terms with i_k's still intact), we get the alternating matching polynomial. To be more precise, $M_{-}(G) = |O(G)| - I(G)$, where $I(G)$ are the terms of $|O(G)|$ containing an i_k. This result is basically identical to a similar one reached by Wahid, the only difference being that he uses square root signs to delineate the unwanted terms. For a careful proof, see [5].

The problem with this "quicker" method of computing the matching polynomial is that computing determinants of matrices with many variables is highly innefficient, computationally. So we have essentially moved the complexity from the creation of a branching tree of potentially many subgraphs to the computation of a single, very complex determinant.

Conclusion

Unfortunately, neither algorithm is particularly computationally efficient in a general case. So the search for a more efficient algorithm to compute the matching polynomial continues. Though we did not succeed in improving efficiency, hopefully we at least shed some more light on the fascinating connection between determinants and the matching polynomial.

Bibliography

[1] E.J. Farrell. *An Introduction to Matching Polynomials*. Journal of Combinatorial Theory, Series B Vol. 27 (1979) 75-86.

[2] E.J. Farrell and S. A. Wahid. *Matching Polynomials: A Matrix Approach and its Applications*. Journal of the Franklin Institute Vol. 322, Iss. 1 (1987) 13-21.

[3] E.J. Farrell and S. A. Wahid. *D-Graphs 1; An Introduction to Graphs Whose Matching Polynomials are Determinants of Matrices*. Bulletin of the Institute of Combinatorics and its Applications Vol. 15 (1995) 81-86.

[4] C.D. Godsil. *On the Theory of the Matching Polynomial*. Journal of Graph Theory Vol. 5, Iss. 2 (1981) 137-145.

[5] S.A. Wahid, *The Determinant of Matching Matrix in the Evaluation of the Matching Polynomial*. Bulletin of the Institute of Combinatorics and its Applications Vol. 36 (2002) 62-72.